Mass formula for strange baryons in large N_c QCD versus quark model

Claude Semay [a],[∗] Fabien Buisseret [a],[†] and Florica Stancu [b][‡]

[a] Groupe de Physique Nucléaire Théorique,

Université de Mons-Hainaut, Académie universitaire Wallonie-Bruxelles,

Place du Parc 20, B-7000 Mons, Belgium.

[b] University of Liège, Institute of Physics B5,

Sart Tilman, B-4000 Liège 1, Belgium.

(Dated: February 1, 2008)

Abstract

A previous work establishing a connection between a quark model, with relativistic kinematics and a Y-confinement plus one gluon exchange, and the $1/N_c$ expansion mass formula is extended to strange baryons. Both methods predict values for the SU(3)-breaking mass terms which are in good agreement with each other. Strange and nonstrange baryons are shown to exhibit Regge trajectories with an equal slope, but with an intercept depending on the strangeness. Both approaches agree on the value of the slope and of the intercept and on the existence of a single good quantum number labeling the baryons within a given Regge trajectory.

[∗]E-mail: claude.semay@umh.ac.be

[†]E-mail: fabien.buisseret@umh.ac.be

[‡]E-mail: fstancu@ulg.ac.be

Typeset by REVTEX   1

I. INTRODUCTION

In Ref. [1] we have made a first attempt to establish a connection between quark model results for baryon masses and the $1/N_c$ expansion mass formula. So far we have considered nonstrange baryons only. The purpose of the present study is to extend the previous work to strange baryons.

The standard approach to baryon spectroscopy is the constituent quark model where the results are model dependent. The states are classified according to SU(6) symmetry. The phenomenological analysis suggested that the baryons can be grouped into excitation bands $N = 0,1,2,...$ each band containing at least one SU(6) multiplet. The hyperfine interaction breaks the SU(6) symmetry. The introduction of N as a good quantum number for a Hamiltonian with a linear Y-junction confinement is quite natural for nonstrange baryons [1]. We shall show that, even for strange baryons, N is a good classification number within the same quark model. The problem has already been discussed qualitatively in a nonrelativistic model with a quadratic two-body confinement [2]. Here we provide a quantitative proof of the role of the strange quark mass $m_s \neq m_{u,d}$ within a semi-relativistic model with a realistic confinement.

The $1/N_c$ expansion method [3, 4] offers an alternative, model independent way, to study baryon spectroscopy in a systematic way. The method stems from the discovery that in the limit $N_c \to \infty$, where N_c is the number of colors, QCD possesses an exact contracted SU($2N_f$) symmetry [5, 6] where N_f is the number of flavors. This symmetry is only approximate for finite N_c so that corrections have to be added in powers of $1/N_c$. Here we discuss the case $N_f = 3$. Thus, SU(6) is a common symmetry for both approaches. The $1/N_c$ expansion method has extensively and successfully been applied to ground state baryons ($N = 0$) [7, 8, 9, 10]. Its applicability to excited states is a subject of current investigations. The most studied bands so far are $N = 1$ and 2.

It is important to compare the two methods. This could bring support to quark model assumptions on one hand and it could help to gain more physical insight into the dynamical coefficients of the $1/N_c$ expansion mass formula on the other hand. The key aspect in this comparison is that one can analyze both the $1/N_c$ expansion and quark model results in terms of N. The paper is organized as follows. The mass formula used in the $1/N_c$ expansion for strange baryons is introduced in the next section. Section III gives a corresponding

mass formula obtained from a Hamiltonian quark model of spinless Salpeter type, where the confinement is a Y-junction flux tubes and where the Coulomb-type one gluon exchange and the quark self-energy contributions are added perturbatively. There, we analytically prove that the classification of light baryons, containing u, d, s quarks, is still possible in terms of a quantum number N, representing units of excitation, like in a harmonic oscillator picture. Section IV is devoted to the comparison of the results derived from the $1/N_c$ expansion on one hand and from the quark model on the other hand. A discussion on Regge trajectories emerging from the quark model mass formula is also given. Conclusions are finally drawn in Sec. V.

II. STRANGE BARYONS IN LARGE N_c QCD

A. Mass formula

For strange baryons the mass operator in the $1/N_c$ expansion has the general form

$$M = \sum_{i=1} c_i O_i + \sum_{i=1} d_i B_i, \tag{1}$$

where the operators O_i are invariants under SU(6) transformations and the operators B_i explicitly break SU(3)-flavor symmetry. The coefficients c_i and d_i are fitted from the experimental data and encode the quark dynamics. In the case of nonstrange baryons, only the operators O_i contribute while B_i are defined such as their expectation values are zero. Thus in Ref. [1], devoted to nonstrange baryons, only the first term of Eq. (1) entered the discussion. Presently we focus on the second term. The mass m_s of the strange quark breaks SU(3)-flavor explicitly. The operators B_i and coefficients d_i are used to construct a mass shift ΔM_s introduced below.

In Eq. (1) the sum over i is finite and in practice it includes the most dominant operators. The building blocks of O_i and B_i are the SU(6) generators: S_i (i = 1,2,3) acting on spin and forming an su(2) subalgebra, T^a (a = 1,...,8) acting on flavor and forming an su(3) subalgebra, and G^{ia} acting both on spin and flavor subspaces. For orbitally excited states, also the components ℓ_i of the angular momentum, as generators of SO(3), and the tensor operator ℓ^{ij} (see *e.g.* Ref. [11]) are necessary to build O_i and B_i. Examples of O_i and B_i can be found in Refs. [11, 12, 13, 14]. Each operator O_i or B_i carries an explicit factor of

$1/N_c^{n-1}$ resulting from the power counting rules [4], where n represents the minimum of gluon exchanges to generate the operator. In the matrix elements there are also compensating factors of N_c when one sums coherently over N_c quark lines. In practice it is customary to drop higher order corrections of order $1/N_c^2$.

We assume that each strange quark brings the same contribution ΔM_s to the SU(3)-breaking terms in the mass formula. To make a comparison between the $1/N_c$ expansion and the quark model results we define ΔM_s to satisfy the relation

$$n_s\, \Delta M_s = \sum_{i=1} d_i B_i \qquad (2)$$

where n_s is the number of strange quarks in a baryon.

Previous studies have indicated that, to a very good approximation, one can apply the $1/N_c$ expansion mass formula to a specific band and neglect interband mixing (see *e.g.* Refs. [11, 12, 13, 14]), so that we can make a comparison of the results of both approches, band by band.

For $N = 0$, 1, and 3, we adopt the values of ΔM_s provided by Ref. [15] and exhibited below in Table III. Actually, for $N = 3$, the only available values are from this reference, where they have been calculated in an approximate way. The large error bars suggest that this band must be more precisely reanalyzed in the large N_c approach. The fit has been made on the $[70, 3^-]$ multiplet, which is the lowest among the eight multiplets contained in this band [16].

For $N = 4$, Table I of Ref. [14] straightforwardly gives $\Delta M_s|_{N=4} = b_1 = 110 \pm 67$ MeV. Due to the scarcity of data, there the analysis was restricted to $[56, 4^+]$, also the lowest SU(6) multiplet, out of 17 theoretically found multiplets, in this band [17].

For $N = 2$ the data are richer. In the next subsection we show how to estimate the mass shift ΔM_s for $N = 2$, by using values of d_i determined in previous $1/N_c$ expansion studies of several SU(6) multiplets.

B. The SU(3)-breaking in the $N = 2$ band

Here we discuss details of the SU(3)-breaking for orbitally excited baryons belonging to the $[56, 2^+]$, $[70, 0^+]$ or $[70, 2^+]$ multiplets of the $N = 2$ band. The $[20, 1^+]$ multiplet of the $N = 2$ band is not physically relevant. We combine the large N_c results obtained for $[56, 2^+]$

in Ref. [13] and for $[70, \ell^+]$ ($\ell = 0, 2$) in Ref. [11]. For the $[56, 2^+]$ multiplet the situation is simple. There are 18 strange resonances in this sector. The analysis of Ref. [13] gives $\Delta M_s|_{[56,2^+]} = 206 \pm 18$ MeV.

For $[70, \ell^+]$ the situation is more complicated. The SU(3)-breaking can be measured by means of Eq. (2) where in the right hand side we replace B_i by their expectation values. In this case there are two dominant operators, B_1 and B_2, and we have $d_1 = 365 \pm 169$ MeV and $d_2 = -293 \pm 54$ MeV from Ref. [11]. Then for a given baryon i we have

$$\Delta M_s(i) = \frac{d_1 B_1 + d_2 B_2}{n_s}. \tag{3}$$

The $[70, \ell^+]$ strange resonances of given isospin I and strangeness $\mathcal{S} = -n_s$ are shown in Table I together with the expectation values of B_1 and B_2 (multiplied by $\sqrt{3}$), the values of $\Delta M_s(i)$ obtained from Eq. (3) and the multiplicity $\nu(i)$ of each baryon. The multiplicity represents the total number of states of distinct total angular momentum, obtained from $\ell = 0, 2$, but having the same values for I, $\mathcal{S}$, B_1, B_2 and $\Delta M_s(i)$.

We calculate an average over all strange resonances belonging to $[56, 2^+]$ or $[70, \ell^+]$ defined as

$$\Delta M_s = \frac{\sum_i \nu(i) \Delta M_s(i)}{\sum_i \nu(i)}. \tag{4}$$

If the average is restricted to members of the $[70, \ell^+]$ multiplet, by using Table I we get

$$\Delta M_s|_{[70,\ell^+]} = 77 \pm 61 \text{ MeV}. \tag{5}$$

The total average including $[56, 2^+]$ with $\Delta M_s(i) = 206 \pm 18$ MeV, $\nu(i) = 18$ and $[70, \ell^+]$ with $\Delta M_s(i) = 77 \pm 61$ MeV, $\nu(i) = 36$ is

$$\Delta M_s|_{N=2} = 120 \pm 47 \text{ MeV}. \tag{6}$$

The error bars on $\Delta M_s(i)$ and on ΔM_s result from error bars on d_i's. The error bars on $\Delta M_s(i)$ were defined as the quadrature of two uncorrelated errors.

III. QUARK MODEL FOR STRANGE BARYONS

A. The Hamiltonian

The potential model used to describe strange baryons is nearly identical to the one which was proposed in Ref. [1]. We refer the reader to this reference for a detailed discussion

TABLE I: The strange baryons belonging to the $[70, \ell^+]$ multiplet with their isospin I and strangeness $\mathcal{S}$. Columns 4 and 5 indicate the expectation values of the operators $B_1\sqrt{3}$ and $B_2\sqrt{3}$ respectively obtained from Tables I, II and V of Ref. [11]. Column 6 gives $\Delta M_s(i)$ obtained from Eq. (3). Column 7 gives the multiplicity of the states exhibited in Column 1 (see text).

Baryon	I	$\mathcal{S}$	$B_1\sqrt{3}$	$B_2\sqrt{3}$	$\Delta M_s(i)$	$\nu(i)$
$^2\Lambda(70, \ell^+)$	0	-1	$-1/2$	-1	64 ± 58	3
$^2\Sigma(70, \ell^+)$	1	-1	$-1/2$	-1	64 ± 58	3
$^2\Xi(70, \ell^+)$	1/2	-2	-1	-2	64 ± 58	3
$^4\Lambda(70, \ell^+)$	0	-1	0	$-3/2$	254 ± 47	5
$^4\Sigma(70, \ell^+)$	1	-1	-1	$-1/2$	-126 ± 99	5
$^4\Xi(70, \ell^+)$	1/2	-2	$-1/2$	$-5/2$	159 ± 46	5
$^2\Sigma'(70, \ell^+)$	1	-1	$-1/2$	-1	64 ± 58	3
$^2\Xi'(70, \ell^+)$	1/2	-2	-1	-2	64 ± 58	3
$^2\Omega(70, \ell^+)$	0	-3	$-3/2$	-3	64 ± 58	3
$^2\Lambda'(70, \ell^+)$	0	-1	$-1/2$	-1	64 ± 58	3

of the Hamiltonian, but we nevertheless recall its main physical content in order to be self-contained.

A baryon, viewed as a bound state of three quarks, can be described in a first approximation by the following spinless Salpeter Hamiltonian

$$H = \sum_{i=1}^{3} \sqrt{\vec{p}_i^2 + m_i^2} + V_Y, \tag{7}$$

where m_i is the current mass of the quark i, and where V_Y is the confining interaction potential. Studies based on both the flux tube model [18] and lattice QCD [19] suggest that the Y-junction is the correct configuration for the flux tubes in baryons: A flux tube, with energy density (or string tension) a, starts from each quark and the three tubes meet at the Toricelli point of the triangle formed by the three quarks. This last point, denoted by $\vec{x}_T$, minimizes the sum of the flux tube lengths. As $\vec{x}_T$ is a complicated function of the quark positions, it is useful for our purpose to approximate the genuine confining potential by the

more easily computable expression

$$V_Y = a \sum_{i=1}^{3} \left| \vec{x}_i - \vec{R} \right|, \tag{8}$$

where $\vec{x}_i$ is the position of the quark i is and $\vec{R}$ the position of the center of mass. The replacement of the Toricelli point by the center of mass leads to a simplified confining potential which actually overestimates the potential energy of the genuine Y-junction by about 5% in most cases [20]. The accuracy of the formula (8) is thus rather satisfactory and can be improved by a simple rescaling of a [20]. In Sec. IV A, we shall show how to rescale it correctly. Let us note that, in Ref. [1], we used a more complex and accurate approximate form for V_Y [see Eq. (10) of this reference]. But, as we shall see later on, the inclusion of strange quarks significantly increases the difficulty of the analytic work, so that we have to restrict ourselves to the more tractable potential (8) in order to obtain closed formulas.

Considering only the confining energy is sufficient to understand the Regge trajectories of light baryons, but not to reproduce the absolute value of their masses. Other contributions are actually needed to lower the mass spectrum; we shall include them perturbatively. The most widely used correction to the Hamiltonian (7) is a Coulomb-like interaction of the form

$$\Delta H_{\text{oge}} = -\frac{2}{3} \alpha_S \sum_{i<j} \frac{1}{|\vec{x}_i - \vec{x}_j|}, \tag{9}$$

arising from one gluon exchange processes, where α_S is the strong coupling constant, usually assumed to be around 0.4 for light hadrons [21, 22].

The other interesting contribution to the mass, which can be added perturbatively as well, is the quark-self energy. Recently, it was shown that the quark self-energy, which is created by the color magnetic moment of a quark propagating through the vacuum background field, adds a negative contribution to the hadron masses [23]. The quark self-energy contribution for a baryon is given by [23]

$$\Delta H_{\text{qse}} = -\frac{fa}{2\pi} \sum_i \frac{\eta(m_i/\delta)}{\mu_i}. \tag{10}$$

The factors f and δ have been computed in lattice QCD studies. First quenched calculations gave $f = 4$ [24]. A more recent unquenched study gives $f = 3$ [25]. Since its value is still a matter of research, it may presently be assumed that $f \in [3, 4]$. Moreover, the value of the gluonic correlation length, denoted as δ, is located in the interval $[1.0, 1.3]$ GeV [24, 25].

7

The function $\eta(\epsilon)$ is analytically known and reads [23]

$$
\eta(\epsilon) = \begin{cases} \left[\dfrac{-3\epsilon^2}{(1-\epsilon^2)^{5/2}} \ln\left(\dfrac{1+\sqrt{1-\epsilon^2}}{\epsilon} \right) + \dfrac{1+2\epsilon^2}{(1-\epsilon^2)^2} \right] & (\epsilon < 1) \\[4mm] \left[\dfrac{-3\epsilon^2}{(\epsilon^2-1)^{5/2}} \arctan\left(\sqrt{\epsilon^2-1} \right) + \dfrac{1+2\epsilon^2}{(1-\epsilon^2)^2} \right] & (\epsilon > 1). \end{cases} \tag{11}
$$

By definition $\eta(0) = 1$, and then quickly decreases for higher values of ϵ, *i.e.* for heavy quarks. It can be checked that, as long as $\epsilon \lesssim 0.3$, $\eta(\epsilon)$ is approximated with a reasonable accuracy by

$$
\hat{\eta}(\epsilon) = 1 - \beta\,\epsilon^2 \tag{12}
$$

with $\beta \approx 2.85$. Moreover, this approximation is especially good for values of $\epsilon = m/\delta$ corresponding to the strange quark mass scale. Consequently, the replacement of η by $\hat{\eta}$ is justified and it is sufficient for our purpose. Finally, μ_i is the dynamical mass of the quark i, defined as the expectation value [23]

$$
\mu_i = \left\langle \sqrt{\vec{p}_i^{\,2} + m_i^2} \right\rangle. \tag{13}
$$

Thus μ_i is state-dependent, since it is computed by averaging the kinetic energy of quark i with the wave function of the unperturbed spinless Salpeter Hamiltonian (7).

B. General formulas

In this work, we are mainly interested in analytical results, needed in a straightforward comparison with the large N_c mass formula. To this aim, let us now introduce auxiliary fields [23], in order to get rid of the square roots appearing in the Hamiltonian (7). We obtain

$$
H(\mu_i, \nu_j) = \sum_{j=1}^{3} \left[\frac{\vec{p}_j^{\,2} + m_j^2}{2\mu_j} + \frac{\mu_j}{2} \right] + \sum_{j=1}^{3} \left[\frac{a^2(\vec{x}_j - \vec{R})^2}{2\nu_j} + \frac{\nu_j}{2} \right]. \tag{14}
$$

The auxiliary fields, denoted by μ_i and ν_j, are, strictly speaking, operators. Although being formally simpler, $H(\mu_i, \nu_j)$ is equivalent to H up to the elimination of the auxiliary fields thanks to the constraints

$$
\delta_{\mu_i} H(\mu_i, \nu_j) = 0 \;\Rightarrow\; \hat{\mu}_i = \sqrt{\vec{p}_i^{\,2} + m_i^2}, \tag{15a}
$$

$$
\delta_{\nu_j} H(\mu_i, \nu_j) = 0 \;\Rightarrow\; \hat{\nu}_i = a|\vec{x}_i - \vec{R}|, \tag{15b}
$$

It is worth mentioning that $\langle \hat{\mu}_i \rangle$ is nothing else than the dynamical quark mass introduced in Eq. (13), and that $\langle \hat{\nu}_i \rangle$ is the energy of the flux tube linking the quark i to the center of mass. Although the auxiliary fields are operators, the calculations are considerably simplified if one considers them as real numbers. They are finally fixed in order to minimize the baryon mass [26], and the extremal values of μ_i and ν_j, denoted by $\mu_{i,0}$ and $\nu_{j,0}$, are logically close to $\langle \hat{\mu}_i \rangle$ and $\langle \hat{\nu}_j \rangle$ respectively.

In Ref. [27], it has been shown that the eigenvalues of a Hamiltonian of the form (14) can be analytically found by making an appropriate change of variables, the quark coordinates $\vec{x}_i = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ being replaced by new coordinates $\vec{x}'_k = \left\{ \vec{R}, \vec{\xi}, \vec{\eta} \right\}$. The center of mass is defined as

$$\vec{R} = \frac{\mu_1 \vec{x}_1 + \mu_2 \vec{x}_2 + \mu_3 \vec{x}_3}{\mu_t}, \tag{16}$$

with $\mu_t = \mu_1 + \mu_2 + \mu_3$ and $\{\vec{\xi}, \vec{\eta}\}$ being the two relative coordinates. As we consider only light baryons, composed of n quarks (n denoting both u or d quarks) and s quarks, the general formulas obtained in Ref. [27] can be simplified in the case where two quarks are of the same mass. Let us set $m_1 = m_2 = m$. By symmetry, we have then $\mu_1 = \mu_2 = \mu$ and $\nu_1 = \nu_2 = \nu$. The mass spectrum of the Hamiltonian (14) is given in this case by [27]

$$M(\mu, \mu_3, \nu, \nu_3) = \omega_\xi(N_\xi + 3/2) + \omega_\eta(N_\eta + 3/2) + \mu + \nu + \frac{\mu_3 + \nu_3}{2} + \frac{m^2}{\mu} + \frac{m_3^2}{2\mu_3}, \tag{17}$$

where

$$\omega_\xi = \frac{a}{\sqrt{\mu \nu}}, \quad \omega_\eta = \frac{a}{\sqrt{2\mu + \mu_3}} \sqrt{\frac{\mu_3}{\mu \nu} + 2\frac{\mu}{\mu_3 \nu_3}}. \tag{18}$$

The integers $N_{\xi/\eta}$ are given by $2n_{\xi/\eta} + \ell_{\xi/\eta}$, where $n_{\xi/\eta}$ and $\ell_{\xi/\eta}$ are respectively the radial and orbital quantum numbers relative to the variable $\vec{\xi}$ or $\vec{\eta}$ respectively. One can also easily check that

$$\left\langle \vec{\xi}^2 \right\rangle = \frac{N_\xi + 3/2}{\phi \, \omega_\xi}, \quad \left\langle \vec{\eta}^2 \right\rangle = \frac{N_\eta + 3/2}{\phi \, \omega_\eta}, \tag{19}$$

with

$$\phi = \sqrt{\frac{\mu^2 \mu_3}{2\mu + \mu_3}}. \tag{20}$$

These last identities provide relevant informations about the structure of the baryons, since

$$\left\langle \vec{X}^2 \right\rangle = \left\langle (\vec{x}_1 - \vec{x}_2)^2 \right\rangle = \sqrt{\frac{4\mu_3}{2\mu + \mu_3}} \left\langle \vec{\xi}^2 \right\rangle, \tag{21}$$

$$\left\langle \vec{Y}^2 \right\rangle = \left\langle \left(\frac{\vec{x}_1 + \vec{x}_2}{2} - \vec{x}_3 \right)^2 \right\rangle = \sqrt{\frac{2\mu + \mu_3}{4\mu_3}} \left\langle \vec{\eta}^2 \right\rangle. \tag{22}$$

Moreover, by symmetry, we can assume the following equality

$$\langle(\vec{x}_1 - \vec{x}_3)^2\rangle = \langle(\vec{x}_2 - \vec{x}_3)^2\rangle \approx \frac{\langle\vec{X}^2\rangle}{4} + \langle\vec{Y}^2\rangle, \tag{23}$$

which will be useful in the computation of the one gluon exchange contribution.

The auxiliary fields appearing in the mass formula (17) have to be eliminated by imposing the constraints $\partial_{\mu_i} M(\mu, \mu_3, \nu, \nu_3) = 0$ and $\partial_{\nu_i} M(\mu, \mu_3, \nu, \nu_3) = 0$. This cannot be done exactly in an analytical way, but, as we shall show in the following, solutions can be found by working at the lowest order in m_i^2. The case $n_s = 0$ has been completely treated in Ref. [1]. As in this last work, we shall assume here that $m_n = 0$.

C.   The case $n_s = 3$

Let us begin by the most symmetric case, that is the case of a baryon formed of three strange quarks (the Ω family). Then, we have $m = m_3 = m_s$, and thus $\mu = \mu_3 = \mu_s$ and $\nu = \nu_3 = \nu_s$ by symmetry. Equation (17) becomes

$$M(\mu_s, \nu_s) = \frac{a}{\sqrt{\mu_s \nu_s}}(N + 3) + \frac{3}{2}\left(\mu_s + \nu_s + \frac{m_s^2}{\mu_s}\right), \tag{24}$$

where $N = N_\xi + N_\eta$. Because of the symmetry of a sss baryon, its mass depends on a single quantum number N only, as for a nnn baryon. This number N is the total number of excitation quanta associated to the Hamiltonian (14). It gives the excitation band of the corresponding eigenstate.

The elimination of ν_s requires that

$$\partial_{\nu_s} M(\mu_s, \nu_s) = 0 \Rightarrow \nu_{s,0} = \left[\frac{a^2(N + 3)^2}{9\mu_s}\right]^{1/3}, \tag{25}$$

and then

$$M(\mu_s) = M(\mu_s, \nu_{s,0}) = \frac{1}{2}\left[\frac{3^4 a^2(N + 3)^2}{\mu_s}\right]^{1/3} + \frac{3}{2}\left(\mu_s + \frac{m_s^2}{\mu_s}\right). \tag{26}$$

The constraint $\partial_{\mu_s} M(\mu_s) = 0$ does not lead to a tractable expression for μ_s, unless a development in powers of m_s^2 is performed. One readily finds that

$$\mu_{s,0} = \mu_0 + \frac{3}{4}\frac{m_s^2}{\mu_0}, \tag{27}$$

with

$$\mu_0 = \sqrt{\frac{a(N + 3)}{3}}, \tag{28}$$

10

satisfies the relation $\partial_{\mu_s} M(\mu_s)|_{\mu_s=\mu_{s,0}} = 0$ at the order m_s^2. Thanks to the relation (27), the mass formula (26) becomes

$$M = 6\mu_0 + \frac{3}{2}\frac{m_s^2}{\mu_0}. \tag{29}$$

The contributions of the one gluon exchange and of the quark self-energy can also be calculated analytically. First, the one gluon exchange mass term is given by

$$\Delta M_{\text{oge}} = -\frac{2}{3}\alpha_s \sum_{i<j} \left\langle \frac{1}{|\vec{x}_i - \vec{x}_j|} \right\rangle \approx -\frac{2\alpha_s}{\sqrt{\langle(\vec{x}_1 - \vec{x}_2)^2\rangle}}, \tag{30}$$

where an obvious symmetry argument has been applied to obtain this last approximate expression. Equation (21) then leads to

$$\Delta M_{\text{oge}} = -\frac{2a\alpha_s}{\sqrt{3}\mu_0}\left(1 + \frac{m_s^2}{4\mu_0^2}\right). \tag{31}$$

The self-energy term (10), together with the approximation (12), is now given by

$$\Delta M_{\text{qse}} = -\frac{3fa}{2\pi}\frac{\hat{\eta}(m_s/\delta)}{\mu_{s,0}} = -\frac{3fa}{2\pi\mu_0}\left[1 - \frac{3m_s^2}{4\mu_0^2} - \frac{\beta m_s^2}{\delta^2}\right]. \tag{32}$$

The total mass for a triply-strange baryon is finally given by the sum $M + \Delta M_{\text{oge}} + \Delta M_{\text{qse}}$.

It is worth mentioning that in the limit $m_s \to 0$, we recover the results of Ref. [1], but the parameter Q of this last reference has to be set equal to 1 –instead of the optimal and very close value of 0.93– in order to take into account our present assumption that the Toricelli point is located at the center of mass. When $m_s \neq 0$, one can wonder about the validity of the Taylor expansion in m_s^2 that we made. The dominant term of Eq. (29) is $6\mu_0$, while the "presumably small" term is $3m_s^2/2\mu_0$. In the worst case, that is for $N = 0$, one has

$$\left.\frac{3m_s^2/2\mu_0}{6\mu_0}\right|_{N=0} = \frac{m_s^2}{4a}. \tag{33}$$

For typical values $m_s = 0.2$ GeV and $a = 0.2$ GeV^2, this ratio is around 0.05. This justifies a posteriori the relevance of such an expansion.

D. The case $n_s = 1$

We turn now to the nns baryons. In this case, we can set $m = 0$ in the formula (17), and replace the index 3 by s, to make clearly the appearance of symbols related to the s quark. The mass formula (17) becomes

$$M(\mu, \mu_s, \nu, \nu_s) = \frac{\omega_\xi + \omega_\eta}{2}(N + 3) + \mu + \nu + \frac{\mu_s + \nu_s}{2} + \frac{m_s^2}{2\mu_s}, \tag{34}$$

with

$$\omega_\xi = \frac{a}{\sqrt{\mu\nu}}, \quad \omega_\eta = \frac{a}{\sqrt{2\mu+\mu_s}}\sqrt{\frac{\mu_s}{\mu\nu}+2\frac{\mu}{\mu_s\nu_s}}. \tag{35}$$

An important simplification has been made in Eq. (34): The term proportional to $\omega_\xi - \omega_\eta$, present only at $N > 0$ and vanishing for nnn and sss baryons, was neglected. This corresponds to the assumption that the integer N is still a good quantum number to classify the asymmetric nns configurations. A numerical resolution of the general formula (17) actually supports this assumption, which is also made in large N_c QCD. This point will be further investigated in Sec. IV B.

The four auxiliary fields appearing in the mass formula (34) can be eliminated by solving simultaneously the four constraints

$$\partial_\mu M(\mu, \mu_s, \nu, \nu_s) = 0, \quad \partial_{\mu_s} M(\mu, \mu_s, \nu, \nu_s) = 0,$$
$$\partial_\nu M(\mu, \mu_s, \nu, \nu_s) = 0, \quad \partial_{\nu_s} M(\mu, \mu_s, \nu, \nu_s) = 0. \tag{36}$$

After some algebra, a solution can be found by working at the order m_s^2, as we did in the previous section for the case $n_s = 3$ (and as we shall do in the rest of this paper). In the following, to simplify the notations, we will write μ (μ_s) for the optimal value of the dynamical mass of the n (s) quark. We find

$$\mu = \mu_0 + \frac{11}{156}\frac{m_s^2}{\mu_0}, \quad \nu = \mu_0 + \frac{7}{156}\frac{m_s^2}{\mu_0},$$
$$\mu_s = \mu_0 + \frac{95}{156}\frac{m_s^2}{\mu_0}, \quad \nu_s = \mu_0 - \frac{53}{156}\frac{m_s^2}{\mu_0}, \tag{37}$$

where μ_0 is still defined by Eq. (28). The mass formula (34), in which the auxiliary fields are replaced by the expressions (37), reads

$$M = 6\mu_0 + \frac{1}{2}\frac{m_s^2}{\mu_0}. \tag{38}$$

The contribution of the one gluon exchange term is a little more involved than in the case $n_s = 3$. With the help of relations (21) and (22), it reads

$$\Delta M_{\text{oge}} \approx -\frac{2}{3}\alpha_s \left[\frac{1}{\sqrt{\langle \vec{X}^2 \rangle}} + \frac{2}{\sqrt{\langle \vec{X}^2 \rangle /4 + \langle \vec{Y}^2 \rangle}} \right] = -\frac{2a\alpha_s}{\sqrt{3}\mu_0}\left[1 + \frac{m_s^2}{12\mu_0^2} \right]. \tag{39}$$

Relations (37) defining μ and μ_s allow to write down the contribution of quark self-energy (10). Using again the approximation (12), we obtain

$$\Delta M_{\text{qse}} = -\frac{3fa}{2\pi\mu_0}\left[1 - \frac{m_s^2}{4\mu_0^2} - \frac{\beta m_s^2}{3\delta^2} \right]. \tag{40}$$

12

E. Results for arbitrary n_s

The case $n_s = 2$ is very similar to the case $n_s = 1$. That is why we will not treat it explicitly in this paper. Rather, we give here a summary of the results which are obtained for arbitrary n_s. Let us recall that, when one deals with a baryon made of three massless quarks ($n_s = 0$), we recover the results of Ref. [1] with $Q = 1$, namely

$$\mu_0 = \sqrt{\frac{a(N+3)}{3}}, \tag{41}$$

and a total baryon mass, including one gluon exchange and quark self- energy, given by

$$M_0 = 6\mu_0 - \frac{2a\alpha_s}{\sqrt{3}\mu_0} - \frac{3fa}{2\pi\mu_0}. \tag{42}$$

By looking at the results of Secs. III C and III D, one can deduce that the auxiliary fields μ and μ_s have the following general form

$$\mu = \mu_0 + \frac{11\,n_s}{156}\frac{m_s^2}{\mu_0}, \qquad n_s = 0, 1, 2, \tag{43}$$

$$\mu_s = \mu_0 + \frac{84 + 11\,n_s}{156}\frac{m_s^2}{\mu_0}, \qquad n_s = 1, 2, 3. \tag{44}$$

Moreover, the total baryon mass is given by

$$M = M_0 + n_s\,\Delta M_s, \qquad n_s = 0, 1, 2, 3, \tag{45}$$

where the contribution of the s quarks is

$$\Delta M_s = \left[\frac{1}{2} - \frac{\alpha_s a}{6\sqrt{3}\mu_0^2} + \frac{fa}{2\pi}\left(\frac{3}{4\mu_0^2} + \frac{\beta}{\delta^2}\right)\right]\frac{m_s^2}{\mu_0}. \tag{46}$$

These formulas are only valid at the order m_s^2. We checked that they agree with the explicit calculation in the case $n_s = 2$. So each s quark brings the same contribution to the baryon mass and this contribution depends on m_s.

The eigenvalues of the spinless Salpeter Hamiltonian with the potential (8) have been numerically calculated in order to check the accuracy of the mass formula (45) with $\alpha_s = f = 0$ (one gluon exchange and self-energy are treated as perturbations). For the relevant values of $m/\sqrt{a}$ and N, the relative error is found around 10%.

IV. COMPARISON OF THE TWO APPROACHES

A. SU(3)-breaking mass term

In both the $1/N_c$ expansion and the quark models the baryon mass is affected by an explicit SU(3)-breaking due to the mass difference between nonstrange u, d and strange s quarks. The effect of SU(3)-breaking in the $1/N_c$ expansion mass formula has been estimated in Sec. II through terms including the operators B_i. Obviously, a nonvanishing value of the strange quark mass also requires the quark model mass formula to be modified as in Eq. (45). Then, the corresponding SU(3)-breaking mass terms (46) can be compared to those resulting from Eq. (2). To do this, we have to determine the values of the parameters in the quark model, since the coefficients of the large N_c formula have already been fitted on the experimental data.

First, we recall that the auxiliary field method yields upper bounds of the mass spectrum, as it is shown in Ref. [28]. This artifact can be cured by making a rescaling of the string tension a, so that the Regge slope of nnn baryons is equal to the Regge slope of $n\bar{n}$ mesons [1]. Obtained within the flux tube model, this slope is $2\pi\sigma$, σ being the physical string tension. By looking at the formula (42), we see that a correct rescaling is made by taking $a = \pi\sigma/6$. In Ref. [1], we have shown that a remarkable compatibility between large N_c QCD and quark model results exists for the nonstrange baryon masses, provided we take $\sigma = 0.163$ GeV^2, $\alpha_s = 0.4$, and $f = 3.5$. These are also the values considered in this work, despite the fact that these parameters were obtained with a value $Q = 0.93$ instead of the value $Q = 1$ assumed here (see Sec. III C). However, two extra parameters are present when strange quarks are taken into account. These are m_s and δ. The value $\delta = 1.0$ GeV has already been used in potential models for mesons, in good agreement with the experimental data [23, 29]. We shall thus use it here too. Finally, m_s was fitted to get an optimal agreement between the quark model and the large N_c QCD mass shift at $N = 0$. We actually found $m_s = 0.243$ GeV, which is larger than the PDG value of 95±25 MeV [30]. However, a strange quark mass in the range 0.2-0.3 GeV is quite usual in potential models [31]. All parameters are gathered in Table II.

Following the error analysis of Ref. [1], $\sigma = 0.163 \pm 0.004$ GeV^2, $\alpha_s = 0.4 \pm 0.05$, and $f = 3.5 \pm 0.12$. We know that $\delta \in [1.0, 1.3]$ GeV. If we allow a variation of 10% for the

parameter β, we find an error on m_s around 12 MeV.

TABLE II: Parameters of the model.

$a = \pi\sigma/6$	$\delta = 1.0$ GeV
$\sigma = 0.163$ GeV^2	$\beta = 2.85$
$\alpha_s = 0.4$	$m_n = 0$
$f = 3.5$	$m_s = 0.243$ GeV

A comparison between the mass shift ΔM_s, obtained with the quark model and its large N_c counterpart, is given in Table III for $N = 0, 1, 2, 3, 4$. One can see that the quark model predictions are always located within the error bars of the large N_c results. Except for $N = 3$, the central values of ΔM_s in the large N_c approach are close to the quark model results. Ignoring the large N_c value at $N = 3$, which would require further investigations, as we argued in Sec. II A, one can see that ΔM_s decreases slowly and monotonously with increasing N, in both methods. This suggests that the central value of ΔM_s obtained in Ref. [15] in the $1/N_c$ approach is probably far too small for $N = 3$.

The results of Table III are plotted in Fig. 1 to see more clearly the evolution of the mass shifts with N. Thus, in both approaches, one predicts a mass shift correction term due to SU(3)-breaking which decreases with the excitation energy (or N).

TABLE III: Mass shifts ΔM_s (MeV) given by Eq. (46) with the parameters of Table II for the quark model, compared to large N_c mass shifts for various values of N: $N = 0, 1, 3$ from Ref. [15], $N = 4$ from Ref. [14]; the $N = 2$ case is studied in detail in Sec. II B.

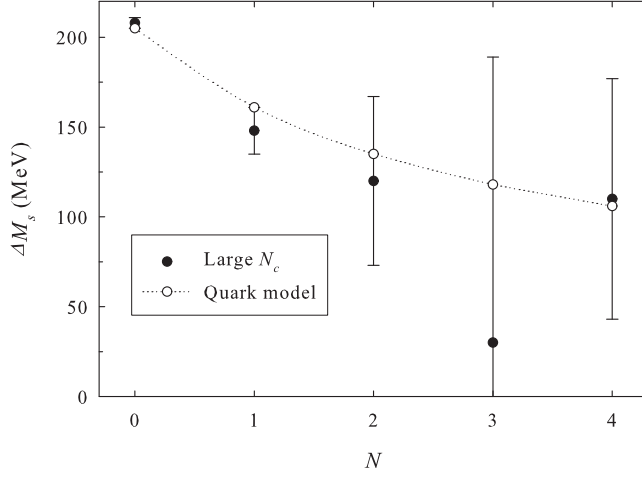| N	Quark model	Large N_c
0	205	208±3
1	161	148±13
2	135	120±47
3	118	30±159
4	106	110±67

FIG. 1: Plot of the results presented in Table III. The quark model predictions for the mass shift ΔM_s (empty circles) are compared to the large N_c data (full circles). A dotted line links the quark model points to guide the eye.

B. The dependence on N

When $n_s = 0$ or 3, the symmetry of the problem leads to a mass formula which depends on $N = N_\xi + N_\eta$ only, with N_ξ and N_η introduced in Sec. III B. When $n_s = 1$ or 2 however, this is not the case. In order to perform explicit calculations, we have assumed that N is still a good quantum number to classify baryon states with one or two strange quarks. This assumption also ensures that the total parity remains $(-1)^N$ in a given band.

To quantitatively support the above considerations, we can now build a quantity to estimate the validity of this approximation for $n_s = 1$ or 2. In this case, the general mass formula (17) must be used. It depends on the auxiliary fields, that we commonly denote here as φ_i, but also on N_ξ and N_η. The value of $M(N_\xi, N_\eta, \varphi_i)$, thus given by Eq. (17), can be computed once σ and m_s are fixed. We choose $\sigma = 0.163\,\mathrm{GeV}^2$ and $m_s = 0.243\,\mathrm{GeV}$ as in the previous section. First, instead of N_ξ and N_η, we work with the quantum numbers N and $N' = N_\xi$, that is to say with the mass formula $M(N', N-N', \varphi_i)$ where $N' = 0, \ldots, N$. Once N and N' are fixed, standard numerical routines allow to minimize the mass with respect to the auxiliary fields. This leads to the optimal values $\varphi_{i,0}$ and finally to $M(N', N-N', \varphi_{i,0})$.

16

Then, we define

$$\delta M(N) = \frac{\max\{M(N', N - N', \varphi_{i,0})\} - \min\{M(N', N - N', \varphi_{i,0})\}}{\left[\sum_{N'=0}^{N} M(N', N - N', \varphi_{i,0})\right] / [N + 1]}, \tag{47}$$

where the maximal and minimal masses are chosen within the set of allowed N' for a given N. δM, which depends only on $m_s/\sqrt{\sigma}$, is a measure of the quality of N as a good quantum number: The more δM is small, the less the value of the mass at a given N depends on the other quantum number N'.

A plot of δM versus N is presented in Fig. 2 for the *nns* and *nss* baryons. By definition, $\delta M(0) = 0$ since the only possibility is $N' = 0$ in this case. Then, it appears that δM increases linearly for $N \geq 1$. Moreover, the values vary very slowly with $m_s/\sqrt{\sigma}$. The key point to observe in this graph is that $\delta M(N \leq 6) \lesssim 3\%$. As no experimental state such that $N > 6$ has so far been observed, we can conclude that the mass formula obtained from the spinless Salpeter Hamiltonian (7) mainly depends on N: A change of N' at a given N only causes a change of the mass which is lower than 3% in all the case that are relevant with regard to current experimental data.
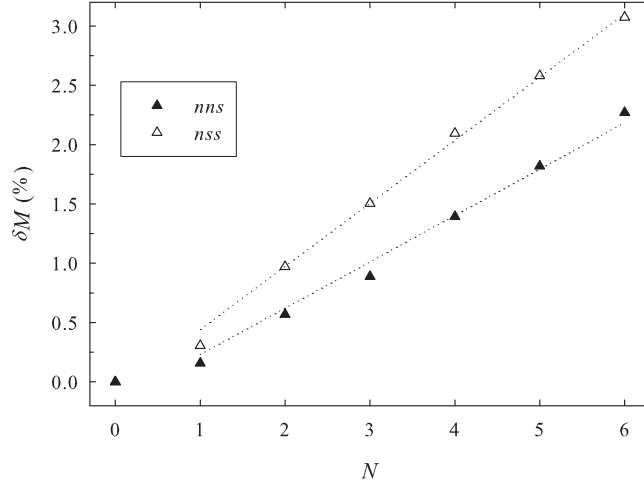

FIG. 2: Plot of δM (%) versus N for *nns* baryons (full triangles) and for *nss* baryons (empty triangles), with the parameters of Table II. Linear fits for the $N > 0$ points are plotted with a dotted line.

Let us note that the previous result is only strictly valid for the mass formula obtained from the Hamiltonian (14). When the eigenstates of the Hamiltonian (7) are computed in

harmonic oscillator bases, it can be seen that these eigenstates contain components from different N bands; thus N is only an approximate good quantum number. Nevertheless, the band mixing is usually small and changes by less than 10% the mass of a state with a dominant component in a given band N.

C. Regge trajectories

Since N appears to be a relevant quantum number to classify light baryons with a good accuracy, it is of interest to study the predictions of large N_c QCD and quark model regarding the Regge trajectories of nonstrange as well as of strange baryons. At the leading order in N, we actually expect that $M^2 \propto N$. Indeed, formula (45) tells us that, at large N,

$$
\begin{aligned}
M^2 &\approx M_0^2 + 2M_0 n_s \Delta M_s \\
&= 2\pi\sigma(N+3) - \frac{4}{\sqrt{3}}\pi\sigma\alpha_s - 3f\sigma + \left[6 + \frac{f\sigma\beta}{\delta^2}\right] n_s m_s^2.
\end{aligned}
\tag{48}
$$

Our particular quark model thus states that baryons should follow Regge trajectories with a common slope, irrespective of the strangeness of the baryons. This feature has also been pointed out in other approaches based on the diquark-quark picture [32, 33]. In Ref. [1] we have already shown that, for nonstrange baryons, M_0^2 was actually equal to $(N_c c_1)^2$, this last quantity being the dominant term in the large N_c mass formula. Moreover, following Ref. [15], the fitted values of c_1 does not change whether or not strange quarks are taken into account. Therefore, the Regge slope of strange and nonstrange baryons is also predicted to be independent of the strangeness in the $1/N_c$ expansion method.

However, the intercept depends on the number of strange quarks. Following the explicit formula (48), it logically increases for larger values of n_s and m_s. Formally, the contribution of strange quarks to the intercept is given in the quark model by $2M_0 \Delta M_s|_{N=0}$, and in the large $1/N_c$ expansion by $2 N_c c_1 \Delta M_s|_{N=0}$. Taking the values of M_0 and c_1 from Ref. [1], and the values of ΔM_s found in this paper, both large N_c QCD and quark model agree on the value of the intercept. The first method leads to 0.361 ± 0.005 GeV^2, while the second one gives 0.355 GeV^2.

The light baryon Regge trajectories are thus predicted to share a common slope, but we expect that they should be separated into parallel straight lines with an intercept depending on the strangeness. Unfortunately, too few experimental data are currently known at large

excitation energies (large N) to check this picture. But, it could be used as an interesting tool to identify strange and nonstrange excited baryons in future experiments.

V. CONCLUSION

The previous work establishing a connection between the quark model and the $1/N_c$ expansion method has been successfully extended to include strange baryons with nonzero mass m_s. A comparison between the SU(3)-breaking terms in the mass formula of the two approaches has been made and we found a good quantitative agreement. The comparison was possible through the introduction of a band quantum number N, customarily used in the baryon classification. While for nonstrange baryons N appears straightforwardly, the inclusion of strange quarks with nonzero mass turned out to be more elaborate. However we have numerically proved that N can be considered as a good quantum number in a realistic quark model with a Y-junction confinement by keeping terms up to order m_s^2 in the Taylor expansion.

---

[1] C. Semay, F. Buisseret, N. Matagne, and Fl. Stancu, Phys. Rev. D **75**, 096001 (2007).

[2] N. Isgur and G. Karl, Phys. Rev. D **19**, 2653 (1979).

[3] G. 't Hooft, Nucl. Phys. **72**, 461 (1974).

[4] E. Witten, Nucl. Phys. B **160**, 57 (1979).

[5] J. L. Gervais and B. Sakita, Phys. Rev. Lett. **52**, 87 (1984); Phys. Rev. D **30**, 1795 (1984).

[6] R. Dashen and A. V. Manohar, Phys. Lett. B **315**, 425 (1993); ibid, 438 (1993) .

[7] R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D **49**, 4713 (1994).

[8] R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D **51**, 3697 (1995).

[9] E. Jenkins, Ann. Rev. Nucl. Part. Sci. **48**, 81 (1998).

[10] E. Jenkins, hep-ph/0111338.

[11] N. Matagne and Fl. Stancu, Phys. Rev. D **74**, 034014 (2006).

[12] J. L. Goity, C. L. Schat, and N. N. Scoccola, Phys. Rev. D **66**, 114014 (2002).

[13] J. L. Goity, C. Schat, and N. N. Scoccola, Phys. Lett. B **564**, 83 (2003).

[14] N. Matagne and Fl. Stancu, Phys. Rev. D **71**, 014010 (2005).

[15] J. L. Goity and N. Matagne, arXiv:0705.3055.

[16] P. Stassart and Fl. Stancu, Phys. Lett. B **269**, 243 (1991).

[17] P. Stassart and Fl. Stancu, Z. Phys. A **359**, 321 (1997).

[18] J. Carlson, J. Kogut, and V. R. Pandharipande, Phys. Rev. D **27**, 233 (1983).

[19] Y. Koma, E.-M. Ilgenfritz, T. Suzuki, and H. Toki, Phys. Rev. D **64**, 014015 (2001).

[20] B. Silvestre-Brac, C. Semay, I. M. Narodetskii, and A. I. Veselov, Eur. Phys. J. C **32**, 385 (2004).

[21] A. M. Badalian and V. L. Morgunov, Phys. Rev. D **60**, 116008 (1999); A. M. Badalian and B. L. G. Bakker, Phys. Rev. D **66**, 034025 (2002).

[22] G. S. Bali, Phys. Rep. **343**, 1 (2001).

[23] Yu. A. Simonov, Phys. Lett. B **515**, 137 (2001); F. Buisseret and C. Semay, Phys. Rev. D **71**, 034019 (2005).

[24] A. Di Giacomo and H. Panagopoulos, Phys. Lett. B **285**, 133 (1992).

[25] A. Di Giacomo and Yu. A. Simonov, Phys. Lett. B **595**, 368 (2004).

[26] C. Semay, B. Silvestre-Brac, and I. M. Narodetskii, Phys. Rev. D **69**, 014003 (2004).

[27] F. Buisseret and C. Semay, Phys. Rev. D **73**, 114011 (2006).

[28] F. Buisseret and V. Mathieu, Eur. Phys. J. A **29**, 343 (2006).

[29] F. Buisseret, Phys. Rev. C **76**, 025206 (2007).

[30] W.-M. Yao *et al.*, J. Phys. G **33**, 1 (2006).

[31] W. Lucha, F. Schöberl, and D. Gromes, Phys. Rep. **200**, 127 (1991).

[32] Yu. A. Simonov, Phys. Lett. B **228**, 413 (1989).

[33] A. Selem and F. Wilczek, hep-ph/0602128.